**Mathematical Induction 1**

1. Prove that $2+2^{2}+2^{3}+…+2^{n}=2^{n+1}-2$.

 Let P(n) be the proposition : $2+2^{2}+2^{3}+…+2^{n}=2^{n+1}-2$

 For P(1), L.H.S.=$ 2=2^{1+1}-2 $=R.H.S., ∴P(1) is true.

 Assume P(k) is true for some natural number k, that is,

 $2+2^{2}+2^{3}+…+2^{k}=2^{k+1}-2$………(1)

 For P(k + 1), $2+2^{2}+2^{3}+…+2^{k}+2^{k+1}=\left(2+2^{2}+2^{3}+…+2^{k}\right)+2^{k+1}$

 $=\left(2^{k+1}-2\right)+2^{k+1}$, by (1)

 $=2×2^{k+1}-2$

 $=2^{k+2}-2$

 ∴ P(k + 1) is true.

 ∴ By the Principle of Mathematical Induction, P(n) is true for all natural numbers, n.

2. Prove by Mathematical Induction: $\sum\_{i=1}^{n-1}i\left(i+1\right)=\frac{n\left(n-1\right)\left(n+1\right)}{3}$, for all integers $n\geq 2$.

 Let P(n) be the proposition : $\sum\_{i=1}^{n-1}i\left(i+1\right)=\frac{n\left(n-1\right)\left(n+1\right)}{3}$, for all integers $n\geq 2$.

 For P(2), L.H.S.=$ 1\left(1+1\right)=2=\frac{2×\left(2-1\right)×\left(1+1\right)}{3}$=R.H.S., ∴P(2) is true.

 Assume P(k) is true for some integerk,$k\geq 2$. that is,$\sum\_{i=1}^{k-1}i\left(i+1\right)=\frac{k\left(k-1\right)\left(k+1\right)}{3}$……(1)

 For P(k + 1), $\sum\_{i=1}^{k}i\left(i+1\right)=\sum\_{i=1}^{k-1}i\left(i+1\right)+k\left(k+1\right)=\frac{k\left(k-1\right)\left(k+1\right)}{3}+k\left(k+1\right)$, by (1)

 $=\frac{k\left(k+1\right)}{3}\left[\left(k-1\right)+3\right]=\frac{k\left(k+1\right)}{3}\left(k+2\right)$

 $=\frac{\left(k+1\right)\left[\left(k+1\right)-1\right]\left[\left(k+1\right)+1\right]}{3}$

 ∴ P(k + 1) is true.

 By the First Principle of Mathematical Induction, P(n) is true for integers $n\geq 2$.

3. Prove $1^{3}+3^{3}+5^{3}+…+ \left(2n+1\right)^{3}= \left(n+1\right)^{2}\left(2n^{2}+4n+1\right)$ by mathematical induction.

 Let $P\left(n\right):1^{3}+3^{3}+5^{3}+…+ \left(2n+1\right)^{3}= \left(n+1\right)^{2}\left(2n^{2}+4n+1\right)$

 For $P\left(1\right)$ , L.H.S. = $1^{3}+3^{3}=28$, R.H.S. = $\left(1+1\right)^{2}\left[2\left(1^{2}\right)+4\left(1\right)+1\right]=28$

 $∴P\left(1\right)$ is true.

 Assume $P\left(k\right)$ is true for some $k\in N$, that is

 $1^{3}+3^{3}+5^{3}+…+ \left(2k+1\right)^{3}= \left(k+1\right)^{2}\left(2k^{2}+4k+1\right)…(1)$

 For $P\left(k+1\right)$,

 $1^{3}+3^{3}+5^{3}+…+ \left(2k+1\right)^{3}+ \left(2k+3\right)^{3}$

 $= \left(k+1\right)^{2}\left(2k^{2}+4k+1\right)+ \left(2k+3\right)^{3}…(2)$ , by (1)

 Now, writing $u=k+1$, then

 $\left(k+1\right)^{2}=u^{2}$

 $2k^{2}+4k+1=2(u-1)^{2}+4\left(u-1\right)+1=2u^{2}-1$

 $\left(2k+3\right)^{3}=\left[2\left(u-1\right)+3\right]^{3}=\left(2u+1\right)^{3}=8u^{3}+12u^{2}+6u+1$

 By (2), $1^{3}+3^{3}+5^{3}+…+ \left(2k+1\right)^{3}+ \left(2k+3\right)^{3}$

 $=u^{2}\left(2u^{2}-1\right)+\left(8u^{3}+12u^{2}+6u+1\right)$

 $= \left(2 u^{4}- u^{2}\right)+ \left(8u^{3}+12u^{2}+6u+1\right)$

 $=2 u^{4}+ 8 u^{3}+11 u^{2}+ 6 u+1$

 $=\left(u+1\right)^{2}\left(2u^{2}+4u+1\right)$ (use Factor Theorem)

 $∴P\left(k+1\right)$ is true.

 By the principle of mathematical induction, $P\left(n\right)$ is true for all $n\in N$.

4. Prove $1^{2}+3^{2}+5^{2}+…+\left(2n-1\right)^{2}=\frac{1}{3}n\left(2n-1\right)\left(2n+1\right)$by mathematical induction.

 Let $P\left(n\right):1^{2}+3^{2}+5^{2}+…+\left(2n-1\right)^{2}=\frac{1}{3}n\left(2n-1\right)\left(2n+1\right)$

 For $P\left(1\right)$ , L.H.S. =$1^{2}=1$ , R.H.S. =$\frac{1}{3}×1×1×3=1$, $∴P(1)$ is true.

 Assume $P\left(k\right)$ is true for some $k\in N$, that is $1^{2}+3^{2}+5^{2}+…+\left(2k-1\right)^{2}=\frac{1}{3}k\left(2k-1\right)\left(2k+1\right)…(1)$

 For $P\left(k+1\right)$,

 $1^{2}+3^{2}+5^{2}+…+\left(2k-1\right)^{2}+\left(2k+1\right)^{2}$

 $=\frac{1}{3}k\left(2k-1\right)\left(2k+1\right)+\left(2k+1\right)^{2}$ , by (1)

 $=\frac{1}{3}\left(2k+1\right)\left[k\left(2k-1\right)+3\left(2k+1\right)\right]$

 $=\frac{1}{3}\left(2k+1\right)\left[2k^{2}+5k+3\right]$

 $=\frac{1}{3}\left(2k+1\right)\left(k+1\right)\left(2k+3\right)$ $=\frac{1}{3}$ $\left(k+1\right)\left[2\left(k+1\right)-1\right]\left[2\left(k+1\right)+1\right]$

 $∴P\left(k+1\right)$ is true.

 By the principle of mathematical induction, $P\left(n\right)$ is true for all $n\in N$.

5. Prove $1^{3}+2^{3}+3^{3}+…+ n^{3}= \left[\frac{n\left(n+1\right)}{2}\right]^{2}$ by mathematical induction.

 Let $P\left(n\right):1^{3}+2^{3}+3^{3}+…+ n^{3}= \left[\frac{n\left(n+1\right)}{2}\right]^{2}$

 For $P\left(1\right)$ , L.H.S. = $1^{3}=1$, R.H.S. = $\left[\frac{1×2}{2}\right]^{2}=1$

 $∴P\left(1\right)$ is true.

 Assume $P\left(k\right)$ is true for some $k\in N$, that is

 $1^{3}+2^{3}+3^{3}+…+ k^{3}= \left[\frac{k\left(k+1\right)}{2}\right]^{2}…(1)$

 For $P\left(k+1\right)$,

 $1^{3}+2^{3}+3^{3}+…+ k^{3}+ \left(k+1\right)^{3}$

 $= \left[\frac{k\left(k+1\right)}{2}\right]^{2}+ \left(k+1\right)^{3}$ , by (1)

 $=\frac{\left(k+1\right)^{2}}{4}\left[k^{2}+4(k+1)\right]$

 $=\frac{\left(k+1\right)^{2}}{4}\left(k+2\right)^{2}$

 $=\left[\frac{\left(k+1\right)\left[\left(k+1\right)+1\right]}{2}\right]^{2}$

 $∴P\left(k+1\right)$ is true.

 By the Principle of mathematical induction, $P\left(n\right)$ is true $∀n\in N$**.**

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